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Semi-implicit representations of surfaces in $\mathbb{P}^3$, resultants and applications☆

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Abstract

In this paper we introduce an intermediate representation of surfaces that we call semi-implicit. We give a general definition in the language of projective complex algebraic geometry, and we begin its systematic study with an effective view-point. Our last section will apply this representation to investigate the intersection of two bi-cubic surfaces; these surfaces are widely used in Computer Aided Geometric Design.

Keywords: Algebraic family of curves; Resultants; Determinantal varieties

1. Introduction

Parametric and implicit representations of surfaces in $\mathbb{R}^3$ offer complementary advantages for the applications in engineering, specially in Computer Aided Geometric Design (CAGD for short). The parametric representation presents the surface as the image of a rational map from $\mathbb{R}^2$ to $\mathbb{R}^3$; this allows fast generation of points on the surface and flexibility for designing. The implicit representation defines an algebraic constraint used to

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determine if a point belongs to a surface S and provides indications to locate it in case it is outside S; implicit representation is also useful for surface blending. Intersection of two surface patches can be done accurately if one patch is given by a parametric representation and the other by an implicit representation. Unfortunately, conversion from one representation to the other is not always possible and when possible it is, in general, difficult and costly.

In this paper we introduce an intermediate representation of surfaces that we call *semi-implicit*. We give a general definition in the language of projective complex algebraic geometry, and we begin its systematic study with an effective view-point. Our last section will apply this representation to investigate the intersection curve (which is of degree 324) of two bi-cubic surfaces which are widely used in CAGD.

Our starting observation is the following: a tensor-product parametric surface can be viewed as the projection S in $\mathbb{P}^3$ of the graph $\mathcal{G}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ of a rational map, whereas our object of study will be the intermediate projection $\mathcal{Z}$ in $\mathbb{P}^1 \times \mathbb{P}^3$. This is a surface of codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^3$ fibred over $\mathbb{P}^1$.

We state formal definitions and expected properties. Although the geometry and representation of $\mathcal{Z}$ can be complicated, we single out the special case of surfaces spanned by a family of determinantal curves, for which we derive useful formulae and algorithms. For that purpose we will use an adapted generalized resultant which provides a compact determinantal representation for the corresponding implicit equation. The fact that we obtain this polynomial equation via a resultant is a guaranty of a good numerical stability of the output and allows reliable approximate computations.

The paper is organized as follows. Section 2 describes our setting, defines formally semi-implicit representations of a reduced surface in $\mathbb{P}^3$, states two basic problems and illustrates them. Section 3 describes the needed algebraic tools, including a generalized resultant, and show how to use them to manipulate these semi-implicit representations. Section 4 is devoted to the application of our results to the study of the intersection of two bi-cubic surface patches: we view such surfaces as families of determinantal curves.

We will always work over the algebraically closed field $\mathbb{C}$, unless specified in the text.

2. Semi-implicit representation of surfaces in $\mathbb{P}^3$

An implicit representation of a surface S in $\mathbb{P}^3$ consists in viewing it as a closed subvariety of $\mathbb{P}^3$, i.e. describing it as the zero locus of a collection of homogeneous polynomials in $\mathbb{C}[x, y, z, w]$. In this section we represent surfaces in $\mathbb{P}^3$ in a different way, as *parametrized* families of *implicitly* represented space curves. We call such a representation a *semi-implicit* representation. It basically consists in viewing a surface $S \subset \mathbb{P}^3$ as the projection on the second factor of a certain closed subvariety $\mathcal{Z}$ of $\mathbb{P}^1 \times \mathbb{P}^3$. We restrict our study to the case of reduced pure dimensional surfaces, i.e. not necessarily irreducible but each component is a surface occurring with multiplicity 1.

Before stating a formal definition, let us recall few facts about space curves. Opposed to a parametrized representation which only exists for a rational curve, an implicit representation may represent any space curve. The more common way to describe implicitly a space curve is to give a (minimal) set of generators, say homogeneous polynomials $g_1(x, y, z, w), \ldots, g_n(x, y, z, w)$; the points of the given curve $\mathcal{C}$ are exactly the common zeros of $g_1, \ldots, g_n$. Of course such a representation of $\mathcal{C}$ is not unique; any set of generators of the ideal $I_{\mathcal{C}} = (g_1, \ldots, g_n) \subset \mathbb{C}[x, y, z, w]$ of the curve is appropriate. It is also possible to give an implicit representation with more details which can be useful in practice but have to be computed: a minimal free resolution of $I_{\mathcal{C}}$ (actually unique up to isomorphism), that is a complex $\mathbf{C}_\bullet$ of free $\mathbb{C}[x, y, z, w]$-modules

$$0 \to \mathbb{C}[x, y, z, w]^{m-n+1} \xrightarrow{\phi_3} \mathbb{C}[x, y, z, w]^m \xrightarrow{\phi_2} \mathbb{C}[x, y, z, w]^n \xrightarrow{\phi_1} \mathbb{C}[x, y, z, w] \quad (1)$$

which is acyclic ($H_i(\mathbf{C}_\bullet) = 0$ for $i > 0$) and such that $H_0(\mathbf{C}_\bullet) = \mathbb{C}[x, y, z, w]/I_{\mathcal{C}}$. In this representation ϕ_1 is a vector $(g_1, \ldots, g_n)$ consisting of n homogeneous minimal generators of $\mathcal{C}$ in $\mathbb{P}^3$, ϕ_2 is a matrix whose columns generate the relations between the g_is and ϕ_3 describes the relations between these relations. This complex is graded since $I_{\mathcal{C}}$ is homogeneous. Remark that the length of the resolution is less than 3 and the last exponent is $m - n + 1$ (note however that any such resolution does not necessarily represents a space curve). This representation yields directly some information on the curve and allows a better control on so-called flat deformations of $\mathcal{C}$ (see Eisenbud, 1994, Chapter 6). For instance if the curve is a complete intersection, i.e. defined by two equations, its minimal free resolutions are given by Koszul complexes, and if the curve is arithmetically Cohen–Macaulay then its minimal free resolutions have a particular structure given by the Hilbert–Burch theorem (see Eisenbud, 1994, Theorem 20.15). We also recall that one can naturally associate with such a space curve of degree d a Chow form and consequently a point in the Chow variety $G(2, d, 4)$, see e.g. Gelfand et al. (1994). For the classification problem see also Galligo et al. (2003).

Definition 2.1. A semi-implicit representation of a (reduced) surface $S \subset \mathbb{P}^3$ is a collection of bi-homogeneous polynomials $F_i(s, t; x, y, z, w)$, $i = 1, \ldots, n$, defining a closed subvariety $\mathcal{Z} \subset \mathbb{P}^1 \times \mathbb{P}^3$ such that its projection on the first factor is surjective and is S on the second factor.

$\mathcal{Z}$ is a finite union of irreducible surfaces, therefore, without loss of generality, we can assume here that $\mathcal{Z}$, and hence S, are irreducible. By Bertini's theorem, this definition implies that the generic fiber of the first projection $\pi_1 : \mathcal{Z} \to \mathbb{P}^1$ is an irreducible curve in $\mathbb{P}^3$; by generic flatness, there exists a Zariski open subset U in $\mathbb{P}^1$ such that for all $u \in U$ the fiber $\mathcal{Z}_u$ is an irreducible curve; in other words S is obtained as a *flat* family of space curves over $U \subset \mathbb{P}^1$ and S is the closure in $\mathbb{P}^3$ of $\pi_2(\mathcal{Z}_{|U})$. As we recalled, a space curve can be represented implicitly by a minimal free resolution. Reducing the open subset U if needed, we can represent our family of space curves over U by a minimal free resolution at the generic point of $\mathbb{P}^1$. In other words we can compute a bi-graded complex of $\mathbb{A}[x, y, z, w]$-modules, where $\mathbb{A} = \mathbb{C}[s, t]$,

$$0 \to \mathbb{A}[x, y, z, w]^{m-n+1} \xrightarrow{\phi_3} \mathbb{A}[x, y, z, w]^m \xrightarrow{\phi_2} \mathbb{A}[x, y, z, w]^n \xrightarrow{\phi_1} \mathbb{A}[x, y, z, w],$$

which is acyclic after tensorizing by $\mathbb{A}_p$ over $\mathbb{C}$ for any $p \in U$. Such a *generic* resolution of our surface (only valid on a dense open subset) can be useful in practice giving some information on the structure of the given surface; we will illustrate this point later with some applications.

Remark 2.2. Asking in Definition 2.1 that the projection of $\mathcal{Z}$ is S on $\mathbb{P}^3$ may be restrictive in some cases. However, assuming only that $\mathcal{Z}$ is a surface in $\mathbb{P}^1 \times \mathbb{P}^3$, the projection of $\mathcal{Z}$ on $\mathbb{P}^3$ is then S plus a finite number of surfaces corresponding to the points $(s_0, t_0) \in \mathbb{P}^1$ such that the $F_i(x, y, z, w; s_0, t_0)$s have a common factor defining an extraneous surface in $\mathbb{P}^3$, that is to say to the points where the fiber of the first projection $\mathcal{Z} \to \mathbb{P}^1$ is not a curve but a surface. Similarly to the terminology used for parametrized representations, such points could be called "base points" of the semi-implicit representation.

The first operations to achieve are to go respectively from a parametrized representation of a surface to a semi-implicit representation, a problem that we call the *semi-implicitization problem*, and to go from a semi-implicit representation to an implicit representation, a problem that we call the *implicitization problem* (note that this terminology also refers classically to the problem of computing an implicit representation from a parametrized representation). Both are naturally elimination problems, and hence can be done using Gröbner basis computations. However, similarly to the classical implicitization problem, one also aims to rely on tools involving only linear algebra routines, as resultants. For instance, if $\mathcal{Z}$, representing a surface S, is a complete intersection defined by two polynomials $F_1(s, t; x, y, z, w)$ and $F_2(s, t; x, y, z, w)$, then the usual resultant (also called Sylvester resultant) allows to compute the implicit equation of S. By a direct computation it follows that the degree of S is $k_1 d_2 + k_2 d_1$. In the next section we will see how this setting can be generalized.

3. Algebraic tools

We present some tools which can be used to manipulate semi-implicit representations of surfaces. The two first sections deal with applications of standard results from elimination theory to our particular settings. The two last sections present some particular semi-implicit representations that can be implicitized by a single determinant computation, as well as general degree formula.

3.1. Gröbner basis

General presentations of the theory and techniques for Gröbner bases can be found in several texts books e.g. Cox et al. (1996), Hoffmann (1989); they are implemented in many computer algebra systems. Basically they allow us to perform most algebraic manipulations on ideals of polynomials with coefficients in computable fields. The drawback of this flexibility is the fact that the execution strongly depends on the input data, therefore the control on the growth of the coefficients or on their precision (when they are known only approximately) is hard to achieve.

3.1.1. From a semi-implicit representation to an implicit representation

We start with a semi-implicit representation given by a collection of bi-homogeneous polynomials $F_i(s, t; x, y, z, w)$, $i = 1, \ldots, n$, as in the Definition 2.1. These polynomials generate a bi-graded ideal I in $\mathbb{C}[s, t; x, y, z, w]$. Because of the hypothesis, the ideal $I \cap \mathbb{C}[x, y, z, w]$ is generated by a polynomial $H(x, y, z, w)$, which can be computed via Gröbner basis techniques. Note that with similar computations the hypothesis can be checked.

3.1.2. From a parametric representation to a semi-implicit representation

We start with a parametric representation given by four bi-homogeneous polynomials $\Phi_j(s,t;u,v)$, $j=1,\ldots,4$. We consider the ideal J in $\mathbb{C}(s,t)[u,v,x,y,z,w]$ generated by $x-\Phi_1(s,t;u,v)$, $y-\Phi_2(s,t;u,v)$, $z-\Phi_3(s,t;u,v)$, $w-\Phi_4(s,t;u,v)$. Via Gröbner basis computations in this ring we get minimal generators of the ideal $J\cap\mathbb{C}(s,t)[x,y,z,w]$ and also a minimal resolution of the corresponding quotient algebra. Finally, we get rid of the denominators and obtain a bi-graded ideal in $\mathbb{C}[s,t;x,y,z,w]$ together with a complex of free modules as in the definition.

3.1.3. An illustrative example

We consider the family of quartics defined by the following parametrization of bi-degree (4; 2) in the variables $(s,t;u,v)$:

$$\begin{aligned}
x &= \Phi_1(s,t,u,v) = uvt^4 - v^2s^4\\
y &= \Phi_2(s,t,u,v) = v2st^3 - 2u^2s^4\\
z &= \Phi_3(s,t,u,v) = uvt^2s^2 - 3v^2s^4\\
w &= \Phi_4(s,t,u,v) = v^2ts^3.
\end{aligned}$$

We eliminate the homogeneous variables (s,t) and get four semi-implicit equations $F_j(u,v;x,y,z,w)$, $j=1,\ldots,4$, of respective degrees $(2,3,3,4)$ in (x,y,z,w). We dehomogenize the coefficients by setting $v=1$, then compute a Gröbner basis for the degree order $x>y>z>w$ and coefficients in the field $\mathbb{Q}(u)$. We get four elements in the basis: $G_j(u;x,y,z,w)$, $j=1,\ldots,4$, of respective degrees $(2,3,3,3)$ in (x,y,z,w) and leading monomials (xz, yz^2, y^2z, y^3). A minimal free resolution is obtained from the relations between the G_js, then the relations between these relations. It is written, with $K=\mathbb{Q}(u)$:

$$\begin{aligned}
0 \to K[x,y,z,w](-6) &\xrightarrow{\phi_3} K[x,y,z,w](-5)\oplus K[x,y,z,w](-4)^3 \xrightarrow{\phi_2} \cdots\\
\cdots &\xrightarrow{\phi_2} K[x,y,z,w](-2)\oplus K[x,y,z,w](-3)^3 \xrightarrow{(G_1,\ldots,G_4)} K[x,y,z,w].
\end{aligned}$$

Then we easily get ride of the denominators. The obtained polynomials G_js are reasonable in this illustrative example (at most 12 monomials and coefficients of size less than 100), but can become quickly huge considering more complicated examples.

3.2. Jouanolou–Lazard's matrix

Let S be a surface represented semi-implicitly. The implicitization problem can be tackled using a known result of elimination theory (see Jouanolou, 1980, and also Lazard, 1977, 1981).

Theorem 3.1. *Let A be a noetherian commutative ring, $n\geq 1$ be a given integer, and $A[\underline{X}] := A[X_1,\ldots,X_n]$, where $X_1,\ldots,X_n$ are indeterminates. We denote $\mathfrak{m} = (X_1,\ldots,X_n)$ the irrelevant ideal. Let $f_1,\ldots,f_r$ be $r\geq n$ homogeneous polynomials in*

$A[\underline{X}]$ of respective degree $d_1 \geq d_2 \geq \cdots \geq d_r \geq 1$. Both the following statements are equivalent:

(1) $\exists\, k \in \mathbb{N}$ *such that* $\mathfrak{m}^k \subset (f_1, \ldots, f_r)$,

(2) *The map of free A-modules* $\oplus_{i=1}^r A[\underline{X}](-d_i)_\nu \xrightarrow{(f_1,\ldots,f_r)} A[\underline{X}]_\nu$ *is of maximal rank* $\binom{\nu+n-1}{n-1}$ *for all* $\nu \geq \delta := d_1 + d_2 + \cdots + d_n - n + 1$.

Remark 3.2. Note that in case $A = \mathbb{C}$ then the first statement is equivalent to saying that the polynomials $f_1, \ldots, f_r$ have no common root in $\mathbb{P}^{n-1}$.

Thus we suppose now that the surface S is semi-implicitly represented by n polynomials $F_i(s, t; x, y, z, w)$, with $i = 1, \ldots, n$, bi-homogeneous in the variables s, t and x, y, z, w of respective degree $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ in s, t.

Proposition 3.3. *With the above notations, let δ be the sum of the two greatest integers in the set $\{d_1, \ldots, d_n\}$ minus 1. Then an implicit equation of S is obtained as the* gcd *of the maximal minors of the map, where* $A = \mathbb{C}[x, y, z, w]$,

$$\begin{aligned} \bigoplus_{i=1}^n A[s,t]_{\delta-d_i} &\to A[s,t]_\delta \\ (g_1, \ldots, g_n) &\mapsto \sum_{i=1}^n g_i F_i. \end{aligned}$$

Notice that the matrices involved in this proposition are in general quite big, and almost never square. Observe also that the maximal minors of this previous map contain the resultants of F_1 and several random linear combinations of the other F_is, with respect to (s, t), yielding the implicit equation of S as their greatest common divisor.

3.3. The determinantal resultant and a special class of surfaces

In this subsection we focus on a particular class of surfaces which admit a special type of semi-implicit representations that we call *determinantal*:

Definition 3.4. A determinantal semi-implicit representation of a surface $S \in \mathbb{P}^3$ is a $n \times (n + 1)$ matrix $\mathtt{M}$ whose entry (i, j) is a bi-homogeneous polynomial $H_{i,j}(s, t; x, y, z, w)$ of degree $d_j - k_i > 0$ in (s, t) and such that the $n \times n$ minors of $\mathtt{M}$ give a semi-implicit representation of S.

It appears that a surface S admitting a determinantal semi-implicit representation is such that the generic curve of this representation is Cohen–Macaulay. This information can be read on the resolution of this generic curve: by the Hilbert–Burch theorem (see Eisenbud, 1994, Theorem 20.15) it is of the form

$$0 \to \oplus_{i=1}^n \mathbb{C}[x, y, z, w](-k_i) \to \oplus_{i=1}^{n+1} \mathbb{C}[x, y, z, w](-d_i) \xrightarrow{(g_1,\ldots,g_{n+1})} \mathbb{C}[x, y, z, w]$$

(observe that we just set the last map ϕ_3 to 0 in (1)).

Our interest in determinantal semi-implicit representations is mainly motivated by two facts. The first fact is that parametrized curves in $\mathbb{P}^3$ of degree less than 3 are always determinantal curves, i.e. they can be implicitly represented by the rank default of a 1×2 or a 2×3 matrix. This is obvious for lines, and also for conics since they are forced to be contained in a plane. Almost all the rational cubics are projectively equivalent to the twisted cubic which is the image of $\mathbb{P}^1 \to \mathbb{P}^3 : (s,t) \mapsto (s^3 : s^2t : st^2 : t^3)$ and can be implicitly represented by the locus of the rank default of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}.$$

The others are the intersection of a cubic surface and a plane, thus are complete intersections. It follows that surfaces parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$ with degree less than 3 in one $\mathbb{P}^1$ can be seen as a family of space curves of degree less than 3, and thus admit, at least on a non-empty subset of $\mathbb{P}^1$, a determinantal semi-implicit representation. Note also that in this case the semi-implicitization process consists in simple linear algebra operations (see Section 4 for a detailed example), precisely the ones just mentioned which are needed to compute an implicit representation of a determinantal curve of degree less than 3 from one of its parametrizations.

The second motivating fact for studying determinantal semi-implicit representations is that the elimination of the parameter (s,t) from such a representation may be done by a single determinant computation, which is the usual resultant in the case where the matrix $\mathtt{M}$ is a 1×2 matrix. We now recall how to construct this determinant from Busé (2004) (where it is called determinantal Sylvester resultant): let $\mathtt{M}$ be a matrix as in Definition 3.4 and set

$$m := \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - \min_{j=1,\dots,n}(k_j) - 1.$$

We consider the resultant-type matrix

$$\bigoplus_{i=1}^{n+1} A[s,t]_{d_i - \min_j(k_j)-1} \to A[s,t]_m \ : \ (g_1,\dots,g_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^{i-1} g_i \Delta_i, \qquad (2)$$

where Δ_i, for $i = 1,\dots,n+1$, denotes the determinant of the matrix $\mathtt{M}$ without its ith column. It appears that this matrix is square if and only if $k_1 = \cdots = k_n$ and we thus denote its determinant *Res* and call it the determinantal resultant of $\mathtt{M}$. It is a homogeneous polynomial in x, y, z, w satisfying the resultant-type property: for any given point $(x,y,z,w) \in \mathbb{P}^3$ we have

$$\begin{aligned} Res(x,y,z,w) = 0 &\Leftrightarrow \exists (s,t) \in \mathbb{P}^1 : \operatorname{rank}(M(s,t;x,y,z,w)) < n \\ &\Leftrightarrow \exists (s,t) \in \mathbb{P}^1 : \Delta_i(s,t;x,y,z,w) = 0 \ \forall i = 1,\dots,n+1. \end{aligned}$$

In the case where all the k_is are not equal, a case that we are not going to encounter hereafter, this determinantal resultant can also be defined and computed as the quotient of two determinants, see Busé (2004) for more detail. The multi-degree of the determinantal resultant is also known: *Res* is homogeneous in the coefficients of the ith column of $\mathtt{M}$ of degree $\sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - d_i$. It can be checked by simple computations using the matrix (2) in the case $k_1 = \cdots = k_n$.

With this tool at hand we can solve easily the implicitization problem for determinantal semi-implicitly represented surfaces.

Theorem 3.5. *Suppose we are given a determinantal semi-implicit representation* M *of a surface S as in Definition 3.4 such that polynomials $H_{i,j}(s,t;x,y,z,w)$ are of positive bi-degree $(\alpha_j; d_j - k_i)$. Then*

$$\left(\sum_{i=1}^{n+1}\alpha_i\right)\left(\sum_{i=1}^{n+1}d_i - \sum_{i=1}^{n}k_i\right) - \sum_{i=1}^{n+1}\alpha_i d_i = \beta\deg(S),$$

where β denotes the degree of the generically finite projection of the semi-implicit representation $\mathcal{Z}$ over S. Moreover an implicit equation of S is provided by the determinantal resultant of M.

Observe that the occurrence of β implies that we obtain an implicit equation of S to the power β through this process. However β generically, in terms of the matrix M, equals one. This leads to the notion of a *proper* semi-implicit representation of S, corresponding to the case $\beta = 1$, similarly to the notion of a proper parametrization of a rational surface.

3.4. General case: Using a free resolution of a semi-implicit representation

In this section we consider the general case of a semi-implicitly represented surface knowing a free resolution. More precisely, let $\mathcal{Z} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a semi-implicit representation of a surface S defined by the bi-homogeneous polynomials $F_1(s,t;x,y,z,w), \ldots, F_{r_1}(s,t;x,y,z,w)$, and assume that we have the following minimal free resolution of the ideal $I_{\mathcal{Z}}$ defined by the F_is (and hence that we know its regularity):

$$0 \to \bigoplus_{i=1}^{r_3} R(-d_{3,i}; -k_{3,i}) \to \bigoplus_{i=1}^{r_2} R(-d_{2,i}; -k_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-d_{1,i}; -k_{1,i}) \to R, \quad (3)$$

where R is the bi-graded ring $\mathbb{C}[x,y,z,w] \otimes_{\mathbb{C}} \mathbb{C}[s,t]$. Let us denote by β the degree of the generically finite projection of $\mathcal{Z}$ onto S, and recall that β equals 1 generically.

Theorem 3.6. *With the above notations, the following equality holds:*

$$\sum_{i=1}^{3}(-1)^i \sum_{j=1}^{r_i} d_{i,j}(k_{i,j} - 1) = \beta\deg(S).$$

Moreover, for all integers ν greater or equal to the regularity of $I_{\mathcal{Z}}$ as a $\mathbb{C}[x,y,z,w]$-module the determinant of the resolution of $I_{\mathcal{Z}}$ taken in degree ν equals an implicit equation H of S to the power β.

Proof. This theorem is a consequence of well-known properties of the direct image of a free resolution, here the direct image of the free resolution of $I_{\mathcal{Z}}$ by the projection $\pi : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^3$ which sends $\mathcal{Z}$ to S (see e.g. Jouanolou (1979), and also Gelfand et al. (1994, Chapter 2, Section 2) where such techniques are used). It follows that the determinant of the following free graded complex of $\mathbb{C}[x, y, z, w]$-modules (we set $A := \mathbb{C}[x, y, z, w]$):

$$0 \to \bigoplus_{i=1}^{r_3} A(-d_{3,i})^{\nu-k_{3,i}+1} \to \bigoplus_{i=1}^{r_2} A(-d_{2,i})^{\nu-k_{2,i}+1}$$
$$\to \bigoplus_{i=1}^{r_1} A(-d_{1,i})^{\nu-k_{1,i}+1} \to A^{\nu+1},$$

obtained by taking the degree ν part in variables s and t of the complex (3), is exactly H^β for all ν greater or equal to the regularity of $I_{\mathcal{Z}}$ (this regularity controls the degree where all the higher cohomology of the terms in (3) vanish). Now we can obtain the degree d of this latter determinant, by definition:

$$\bigotimes_{i=1}^{3} \bigwedge \left(\bigoplus_{j=1}^{r_i} A(-d_{i,j})^{\nu-k_{i,j}+1} \right)^{(-1)^i} \simeq A(d).$$

As d is independent of ν, a straightforward computation of the degree of the left side of the previous equality gives the claimed formula after substituting formally ν by zero. □

4. Application: Representing the intersection curve of two bi-cubic patches

In this section we present an application of the concept of semi-implicit representation: the representation of the intersection curve of two bi-cubic Bézier patches which is a curve of degree 324 in $\mathbb{P}^3$.

A bi-cubic Bézier surface is represented in homogeneous coordinates by:

$$S(s, t; s', t') = \begin{pmatrix} X(s, t; s', t') \\ Y(s, t; s', t') \\ Z(s, t; s', t') \\ T(s, t; s', t') \end{pmatrix} = \sum_{i=0}^{3} \sum_{j=0}^{3} V_{i,j} B_i^3(s, t) B_j^3(s', t'),$$

where $V_{i,j} = (X_{i,j}, Y_{i,j}, Z_{i,j}, T_{i,j})$ are the homogeneous control points, both couples $(s : t)$ and $(s' : t')$ are the homogeneous coordinates of a $\mathbb{P}^1$, and $B_i^3(s, t)$ corresponds to the homogeneous Bernstein polynomial

$$B_i^3(s, t) = \binom{3}{i} s^i (t - s)^{3-i}.$$

In other words, $S(s, t; s', t')$ defines a map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ whose image is a surface. If the polynomials X, Y, Z, T are sufficiently generic in $\mathbb{C}[s, t; s', t']$ this surface is of degree $2 \times 3 \times 3 = 18$. Moreover, it is a family of space curves with parameter $(s' : t') \in U \subset \mathbb{P}^1$, i.e. for each given value (s'_0, t'_0), $S(s, t; s'_0, t'_0)$ parametrizes a cubic Bézier space curve in $\mathbb{P}^3$ that we denote by $\mathcal{C}_{s'_0, t'_0}$. Such a curve is generically (that is except for a finite number of value of s'_0, t'_0) a rational normal curve, that is to say projectively equivalent to the twisted cubic

$$\mathbb{P}^1 \to \mathbb{P}^3 \ : \ (s, t) \mapsto (s^3, s^2 t, s t^2, t^3).$$

It appears that such rational normal curves are implicitly determinantal varieties, and hence we can obtain a semi-implicit determinantal representation of our surface S as follows (this process is what we called the semi-implicitization problem).

First we compute the projective transformation A, whose matrix has entries in $\mathbb{C}[s', t']_3$, which sends the twisted cubic on $\mathcal{C}_{s',t'}$; in matrix notations we have $\mathcal{C} = A(s^3, s^2t, st^2, t^3)^t$. In this way we obtain four polynomials X', Y', Z', T' which are linear forms in x, y, z, w with coefficients homogeneous polynomials in s', t' of degree 9 by

$$\begin{pmatrix} X' \\ Y' \\ Z' \\ T' \end{pmatrix} = \det(A) A^{-1} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

(observe that we have multiplied by $\det(A)$, which does not vanish on U, in order to get rid of the denominators). Consider the 2×3 matrix

$$M = \begin{pmatrix} X' & Y' & Z' \\ Y' & Z' & T' \end{pmatrix} : \mathbb{C}[x, y, z, w][s', t'](-1; -9) \to \mathbb{C}[x, y, z, w][s', t'].$$

Its 2×2 minors `Q1,Q2,Q3` give a semi-implicit representation of S, i.e. for all $(s' : t') \in \mathbb{P}^1$ such that $\det(A) \neq 0$, polynomials `Q1,Q2,Q3` describe a rational normal curve $\mathcal{C}_{s',t'}$ in $\mathbb{P}^3$ which is contained in S. Notice that the determinantal resultant of M with respect to s', t' gives an implicit equation in $\mathbb{P}^3$ of degree 54 (see Theorem 3.5), that is to say three times the degree of S (which is $2 \times 3 \times 3 = 18$).

We now consider another bi-cubic Bézier surface $S'(u, v; u', v')$. Substituting the parametric representation of S' in the semi-implicit representation of S, i.e. in the matrix M, we obtain a matrix graded in the following way:

$$\mathbb{C}[u, v; u', v'][s', t'](-3; -3; -9) \to \mathbb{C}[u, v; u', v'][s', t'].$$

Its determinantal resultant with respect to u', v' yields a condition on s', t' and u, v so that both surfaces intersect. The resultant matrix is a 9×9 matrix, whereas classical use of the Dixon resultant for such a problem (which does not use the geometric property of being determinantal) yields an 18×18 matrix, see Canny and Manocha (1992).

Comments. As we observed, the implicitization process of a parametrized surface via a semi-implicit representation is not sharp, since one obtains three times the surface implicit equation. This is due to the fact that the bi-cubic parametrized surfaces form only a subclass of the set of surfaces defined by a semi-implicit representation of that type, namely given by a matrix of degree $(1, 9)$. A direct consequence is that the determinantal resultant representing the intersection curve of S and S' is of bi-degree $(54; 162)$, and not $(54; 54)$ as expected. However, by symmetry, it is possible to obtain another projection on the same space of bi-degree $(162; 54)$, and then recover the good representing curve.

In an annex, we give a simple (printable) example of computation.

5. Conclusion

Motivated by applications in Computer Aided Design, we started the effective study and classification, of surfaces embedded in $\mathbb{P}^3$ which can be viewed as one-parameter algebraic families of spaces curves. We analyzed the conversion problems, semi-implicitization and implicitization, and treated them with general tools from commutative computer algebra. We singled out the important case of a family of determinantal curves. For that case we applied successfully a generalized resultant, developed by the first author, and obtained nice explicit formulae. Our approach and results can be developed further; here are three directions of investigation.

1. In the determinantal case, the surface is naturally and efficiently represented by the relation matrix. It is worthwhile to take low degree polynomials in u for the entries of the relation matrix in order to provide a class of surfaces with a rich geometry and a compact representation. Once exhaustively described, this will be used to create robust models for the reconstruction problem in CAGD.
2. Algebraic geometers have accumulated a large amount of knowledge and methods which, in general, have a high complexity. Fortunately, most of these results are tractable once restricted to curves in $\mathbb{P}^3$. So, potentially, they could be used to improve our results. An example of such a nice idea, which should be re-interpreted from a computational view-point, is the structure theorem of Buschbaum and Eisenbud for free resolutions, see Buchsbaum and Eisenbud (1974).
3. Another useful tool is the notion of liaison, see Peskine and Szpiro (1974). Let us remark that the generic curve C in our example (Section 3.1.3) is in liaison (i.e. roughly speaking is the complementary) with a couple of non-coplanar lines into a complete intersection curve of degree 6. This last curve is defined by a polynomial of degree 2 and one of degree 3. Therefore the surface S in the example could be described semi-implicitly as a complete intersection minus two families of lines. This fact could be exploited in CAGD for computing surface–surface intersections for bi-quartic splines.

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Annex

Hereafter we present an example involving a family of cubics and a family of conics. We made our computations with the software `Macaulay2`, Grayson and Stillman (1993), using a package[1] providing functions to compute different kinds of resultants.

1 available at http://www-sop.inria.fr/galaad/personnel/Laurent.Buse/m2package.html

The parametric formulation of the family of cubics we chose is given by

$$\begin{aligned} \mathcal{C}_\lambda \ :\ & \mathbb{P}^1 \to \mathbb{P}^3 \\ & (s:t) \mapsto (s^3, s^2t - t^3, \lambda s^2 t + st^2, -s^3 + t^3). \end{aligned}$$

This family of cubics is in fact a determinantal family of cubics; their implicit equations in $\mathbb{P}^3$ are obtained as the 2×2 minors of the matrix

$$\begin{pmatrix} X & X+Y+T & -X\lambda - Y\lambda - T\lambda + Z \\ X+Y+T & -X\lambda - Y\lambda - T\lambda + Z & X+T \end{pmatrix}.$$

The family of conics is depending on a single parameter μ. Its parametric representation is

$$\begin{aligned} \mathcal{D}_\mu \ :\ & \mathbb{P}^1 \to \mathbb{P}^3 \\ & (u:v) \mapsto (uv, \mu v^2 + u^2, \mu v^2, v^2). \end{aligned}$$

It is implicitly a complete intersection given by both equations:

$$-X^2 + YT - ZT,\ -\mu T + Z.$$

In this case, we can consider either $\mathcal{C}_\lambda$ or $\mathcal{D}_\mu$ as the implicit family. If $\mathcal{D}_\mu$ is used as the implicit family, a representation of the intersection curve is then given by a classical Sylvester resultant: it is obtained as the determinant of a 9×9 matrix. If now $\mathcal{C}_\lambda$ is the implicit family, a representation of the intersection curve is given in a more compact way by the following 6×6 matrix

$$\left(\begin{array}{cccccc} -1 & 0 & \lambda & 0 & -\lambda^2 & 0 \\ -\lambda-2 & -1 & 2\lambda & \lambda & -2\lambda^2+1 & -\lambda^2 \\ -\lambda-2\mu-3 & -\lambda-2 & 2\lambda\mu+3\lambda-\mu+1 & 2\lambda & -2\lambda^2\mu-3\lambda^2+2\lambda\mu+2 & -2\lambda^2+1 \\ -\lambda\mu-\lambda-\mu-2 & -\lambda-2\mu-3 & 2\lambda\mu+2\lambda-\mu+1 & 2\lambda\mu+3\lambda-\mu+1 & -2\lambda^2\mu-2\lambda^2+2\lambda\mu+\mu+2 & -2\lambda^2\mu-3\lambda^2+2\lambda\mu+2 \\ -\mu^2-2\mu-1 & -\lambda\mu-\lambda-\mu-2 & \lambda\mu^2+2\lambda\mu-\mu^2+\lambda-\mu & 2\lambda\mu+2\lambda-\mu+1 & -\lambda^2\mu^2-2\lambda^2\mu+2\lambda\mu^2-\lambda^2+2\lambda\mu-\mu^2+\mu+1 & -2\lambda^2\mu-2\lambda^2+2\lambda\mu+\mu+2 \\ 0 & -\mu^2-2\mu-1 & 0 & \lambda\mu^2+2\lambda\mu-\mu^2+\lambda-\mu & 0 & -\lambda^2\mu^2-2\lambda^2\mu+2\lambda\mu^2-\lambda^2+2\lambda\mu-\mu^2+\mu+1 \end{array}\right).$$

Developing its determinant we get:

$$\begin{aligned} &\lambda^6\mu^3 + 3\lambda^6\mu^2 - 3\lambda^5\mu^3 + 3\lambda^6\mu - 6\lambda^5\mu^2 - \lambda^3\mu^4 + \lambda^6 - 3\lambda^5\mu - 4\lambda^4\mu^2 + 4\lambda^3\mu^3 + \\ &10\lambda^2\mu^4 + 6\lambda\mu^5 + \mu^6 - 5\lambda^4\mu + 10\lambda^3\mu^2 + 14\lambda^2\mu^3 - 3\lambda\mu^4 - 3\mu^5 - \lambda^4 + 5\lambda^3\mu + \\ &4\lambda^2\mu^2 - 3\lambda\mu^3 + 4\mu^4 + \lambda^3 - 2\lambda^2\mu + 8\lambda\mu^2 - \mu^3 + \lambda\mu + 2\mu^2 - \lambda + 4\mu + 1. \end{aligned}$$

Observe that the process we just described eliminates a parameter on each surface whereas usually one eliminates two parameters of the same surface in order to compute intersecting points using the parametrization of the other surface. However the knowledge of the intersection of two surfaces in CAGD usually require more than one projection of the intersecting curve since one need to "see" this curve from both surfaces (e.g. to know where is the boundary of an object represented by surface patches). Consequently the choice of the projections used is not very important as soon as the needed information is kept.

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